Price Dispersion: an Evolutionary Approach

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Price Dispersion: an Evolutionary Approach

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Abstract

In many markets it is possible to find rival sellers charging different prices for the same good. Earlier research has explained this phenomenon by demonstrating the existence of dispersed price equilibria when consumers must make use of costly search to discover prices. Taking as a starting point the model of Burdett and Judd *(Econometric, 1983),* this paper, extending evolutionary techniques to a game with nonlinear payoffs and a continuum of strategies, reexamines the question of price dispersion from an evolutionary, disequilibrium perspective. That is, firms and consumers adjust behaviour adaptively in response to current market conditions. We find that dispersed price equilibria are unstable when consumers use a fixed sample size search rule but may be stable when a reservation price rule is used.

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1 Introduction

It is a common experience to find that prices vary between different sellers, giving consumers an incentive to search for low prices. Stigler (1961) introduced the notion of modelling consumer search as repeated random draws from the current distribution of prices. In response Rothschild (1973) set up a challenge. He argued that it was not enough to examine, as had Stigler, the search behaviour of consumers faced with an exogenous distribution of prices. Sellers, presumably, would only charge prices different from those of their competitors if they could make a profit by doing so. To explain price dispersion, economists must show that such price-setting behaviour was a rational response by traders to the search behaviour of consumers, and vice-versa. In game-theoretic terms, it was necessary to find a Nash equilibrium, where each firm and consumer adopted a strategy that was a best reply to the play of all other firms and consumers.

In fact, Diamond (1971) had introduced a model which satisfied Rothschild's conditions. However, the model's main result is usually viewed as a paradox. Diamond was able to show that for any positive search costs, in equilibrium, no consumer would search, and all firms would charge the price that maximised joint-profits. This is clearly a Nash equilibrium: when prices are identical, there is no incentive to search; when there is no search, there is no incentive to cut prices to increase sales. Note that the converse state where all consumers are fully-informed and all firms charge a competitive price cannot be a Nash equilibrium. For positive search costs and with all prices identical, active search is not optimal. Since consumers are not fully informed, firms can raise prices without losing all customers. While those economists raised on the "Law of One Price" might have expected price dispersion to be fragile it was surprising that the collapse was in this direction,

Faced with this challenge, subsequent authors, (a partial list includes Salop and Stiglitz, 1977, 1982; Wilde and Schwartz, 1979; Varian, 1980; Burdett and Judd, 1983; Rob, 1985; Bester, 1988; Wilde, 1992; Benabou, 1993), produced models with dispersed price equilibria. However, there remains an unresolved problem with this earlier literature, that of multiple equilibria. There may be more than one equilibrium at which prices are dispersed, and typically, the joint-profit maximizing outcome found by Diamond remains a Nash equilibrium even in the presence of these others.[1] Selecting between these equilibria is not straightforward. It is easily verified that for strictly positive search costs the joint-profit maximizing outcome is a *strict* Nash equilibrium. That is, any deviation leads to strictly lower payoffs. It cannot therefore be easily dismissed.[2]

The other striking difference about the model of Diamond (1971) is that it is "A Model of Price Adjustment" not of equilibrium. There are several advantages to such

[1] This problem occurs in many different models. See Wilde (1992).

[2] For a formal definition of strictness and its place in the heirarchy of equilibrium refinements, see for example van Damme (1991).

a disequilibrium approach. First, it answers the question of how an economy arrives at equilibrium and why one equilibrium is chosen over another. Second, it allows a different approach to the modelling of consumer search. Some models assume as did Stigler that consumers use a fixed sample size search rule, that is, the consumer's problem is to choose a sample size n. The consumer then collects n prices and then takes the lowest offer. More popular has been the assumption of sequential search, that is, after each price quotation the consumer must decide whether to buy at that price or to obtain a further quotation. However, in both cases the common if implausible assumption is that the consumer knows the distribution of prices before starting searching. Here, just as did Diamond (1971), we can relax this assumption. Consumers do not have to know the distribution of prices in order to determine the optimum level of search effort; it can be learnt from experience or from the experience of other consumers.

More recently, disequilibrium models have been back in fashion under the title of learning and evolution. While this work has up to now mostly been on a very abstract level, there may now be enough theoretical ammunition to analyse the problem of price dispersion. Here we present an dynamic, evolutionary model which is able to select between the multiple equilibria present in these models. Evolutionary models are not new to economics but have not always been well received by economists. In particular, the assumption of evolutionary game theory that agents use a single fixed strategy which determines their rate of reproduction may seem ill-suited to economic applications. However, the translation of models between disciplines is not intended to be over-literal. In human society, the births and deaths are of ideas and strategies, not people. We can assume that at each point in time a population of individuals have to choose between different strategies. The state of the system can be summarised by the proportions of the population playing each strategy. The system changes state as agents change strategies. In fact, in an earlier paper (Hopkins, 1995) it was shown that if one aggregates such learning behaviour across a population, the resulting aggregate dynamic is qualitatively similar to the evolutionary replicator dynamics.[3] The exact dynamics do not have to be specified, rather it is possible, as shown by for example, Nachbar (1990), Friedman (1991), Samuelson and Zhang (1992) and Kandori, Mailath and Rob (1993), to work with wide classes of dynamics, which share certain qualitative properties. While this might not represent the behaviour of perfectly rational agents, it encompasses a wide range of plausible adaptive processes and learning schemes, including some quite sophisticated behaviour.

A further criticism might be that much of the recent work in evolutionary games has been in the context of two-player normal-form games, a context that does not encompass most economic problems. Indeed, the game here is an asymmetric game with many players. These are divided in two distinct groups, buyers and sellers. The payoff of each player depends upon the actions of all other players. The payoffs

[3]Both Hofbauer and Sigmund (1988) and van Damme (1991) present excellent surveys of the field of dynamic evolutionary game theory, including description of the replicator dynamics.

are not linear. Firms can choose from a continuum of prices. It is a situation very far from that of a normal-form game. Nonetheless, it is possible to apply the same evolutionary techniques.

Up to now, evolutionary and learning models have been applied to similar problems by using a discrete approximation of a continuous strategy space (for example, Roth and Erev, 1995). This may be a problem in that, for the many models of dispersed price equilibria, the existence and character of equilibrium depends on the properties of the continuum. Hence we develop learning dynamics on the Hilbert space $\mathcal{L}_2$, where we look at the evolution of the functions describing the distribution of prices. While this does involve some technical difficulties, we are able to show that even when firms can choose from a continuum of prices, it is possible to obtain clear results on the stability of dispersed price equilibria under different assumptions. In particular, we analyse the model of Burdett and Judd (1983), which has the advantage of including differing formulations of consumer search behaviour as special cases. We find that when consumers use a fixed sample size rule dispersed price equilibria are unstable, but when a reservation price rule is used, there exist stable dispersed price equilibria.

2 The Model of Burdett and Judd

In this section we outline the nature of equilibrium in the two different models considered by Burdett and Judd (1983). Obviously full details can be found in the original paper. Burdett and Judd demonstrate how a dispersed price equilibrium can arise without any heterogeneity either amongst firms or consumers. We are concerned with a market for a homogeneous good. For example, the same model of car or computer from a particular manufacturer is often sold by many different outlets, often at different prices. Consumers buy this product only infrequently. The sellers we can think of as a continuum of identical small shops, which buy the good from a wholesaler for a constant cost, which here we assume to be zero. We assume (an average of) μ customers per seller. Firms choose prices in order to maximise profits. There is an upper bound on prices p^*, which can be seen as the profit-maximising price in a monopoly situation. A continuum of consumers are uninformed about which firms charge which prices. They must engage in costly non systematic search in order to obtain price quotations.

We consider two cases, first, where consumers use a fixed sample size rule, and second, where search is sequential. In the first case, the consumer must decide how many quotations to obtain at a constant cost c (the convention is that the first quotation is free). Only once all the n price quotations have arrived can the consumer purchase from the firm that offers the lowest price. Such nonsequential search can be optimal (Morgan and Manning, 1985), and fits the case where a consumer must write away for quotations, or where a number of quotations can be obtained by buying a magazine or newspaper. Sequential search is where a consumer obtains one quotation and then decides whether to take another. The classic result is that when the distribution of prices is known, optimal sequential search takes the form of a reservation price rule. That is, the consumer decides on a target price and continues to search until it is found.

What is common to both forms of search is the possibility of *ex post* heterogeneity of information. That is, while starting out with identical information, some consumers will find a better price than others. This is of course what allows the existence of a dispersed price equilibrium. In particular, a proportion q_1 of consumers know of one price, q_2 have two price quotations and so on. This can arise because either consumers use a fixed sample size rule or if consumers search sequentially, but the search is noisy, for each search made there is a probability q_k of finding k quotations simultaneously. If the distribution of prices is given by the cumulative distribution function $F(p)$, the probability for consumers that a given price p is the lowest that they find with two quotations is $2(1-F(p))$, after three $3(1-F(p))^2$. We continue to assume that there is a continuum of sellers with constant zero marginal cost. Hence, profits for firms are then

$$\pi(p) = p\mu \sum_{k=1}^{\infty} q_k k(1-F(p))^{k-1} \tag{1}$$

where μ is the average measure of consumers per firm. Burdet t and Judd show that the only possible distribution of prices in equilibrium must have continuous support on the interval $(\underline{p}, p^*)$, where $\underline{p}$ is to be determined endogenously. If there were a gap in the distribution on some interval (p_i, p_j), a firm charging p_i could raise its price to fill the gap without losing any customers. If there were a mass point in the distribution at some price p_i, a firm could cut its price from p_i by an arbitrarily small amount and gain a discrete jump in sales,

In fact, Burdett and Judd show that when consumers use a fixed sample size search rule, a dispersed price equilibrium is only possible when a proportion $1 > q_1 > 0$ of consumers buy one quotation, and all others $q_2 = 1 - q_1$ obtain two. In equilibrium, the profit for all firms must be equal. Given (l), and

$$\pi(p^*) = p^*\mu q_1 = \underline{p}\mu\,[q_1 + 2q_2] = \pi(\underline{p}),$$

we can solve for both $\underline{p}$ and $F(p)$. Denoting the equilibrium cumulative distribution function by Φ, we have

$$\Phi = \begin{cases} 0, & p < \underline{p} \\ 1 - \dfrac{p^* - p}{p}\dfrac{q_1}{2(1-q_1)}, & \underline{p} < p \le p^* \\ 1, & p > p^* \end{cases} \tag{2}$$

and

$$\underline{p} = p^* \frac{q_1}{2 - q_1}$$

Is this an equilibrium for consumers? If the price distribution is given by Φ, the difference between the expected price paid by a consumer who searches once and a consumer who searches twice is given by

$$V(q_1) = /p \;\; d@(p) - 2\int p(1-\Phi(p))\, d\Phi(p).$$

This is a continuous function of q_1 with a unique maximum on (O, 1). That is, if c is less than the maximum, there are two values of q_1 such that $V(q_1) = c$, that is, such that consumers are indifferent between their two strategies. If c is higher than the maximum, so that search is not worthwhile, no dispersed price equilibrium exists. Burdett and Judd note however, that another equilibrium exists (irrespective of the value of c). It is the same outcome that Diamond (1971) found. That is, the state with $q_1 = 1$ and, for all firms, $p = p^*$. This is in fact the limiting case of (2) when $q_1 \to 1$. However, $V(q_1) \to -\infty$ as $q_1 \to 1$, so that consumer indifference certainly cannot be maintained in this limit.

Diamond's result suggests that a sequential search rule is not conducive to price dispersion. In general terms, given a uniform cost to each search, a population of rational, maximizing consumers will all settle on the same reservation price and will not buy for more. However, as Burdett and Judd show, such an outcome is not incompatible with price dispersion if search is '(noisy". That is, at each search, a consumer has a possibility of seeing more than one price, the number observed being determined by some exogenous probability distribution. In particular, the probability that k prices are observed is q_k. Now, if consumers share a common search cost c, and hence share a single reservation price $\bar{p}$, no seller will choose a price $p > \bar{p}$. However, the profits for each seller charging a price which is acceptable to consumers, are given by (l), just as in the non-sequential case. However, here the distribution of the q_k is exogenous, and does not arise out of consumer choice. Importantly, this leaves open the possibility that $q_k > 0$ for some $k > 2$. Burdett and Judd demonstrate that for $c > 0$, and $O < q_1 < 1$, the unique equilibrium is a dispersed price equilibrium. That is, there is a continuous distribution of prices on some interval $[\underline{p}, \bar{p}]$.

We can illustrate how using an evolutionary dynamic might help to select between equilibria in these models by looking at the very simple example where sellers have a choice between only two prices $\{\underline{p}, p^*\}$. This is not intended to be realistic but the same intuition drives the result in this case as when seller choose from a continuum of prices. We first assume that consumers search with a fixed sample size rule, and again for sake of simplicity, we assume that they must choose between sampling one and sampling two prices. Their expected costs, if a proportion x of the sellers charge $\underline{p}$ and l–x charge p", will be $x\underline{p}+(1-x)p^*$ in the first case andC+(Z2+2Z(1–z))p+(l $-x)^2p^*$ in the second. If q_1 consumers choose the first option, and $q_2 = 1 - q_1$ the second, then sellers' profits are respectively

$$\pi(\underline{p}) = \underline{p}\mu[q_1 + q_2(2 - x)], \text{or}, \ \pi(p^*) = p^*\mu[q_1 + q_2(1 - x)]. \tag{3}$$

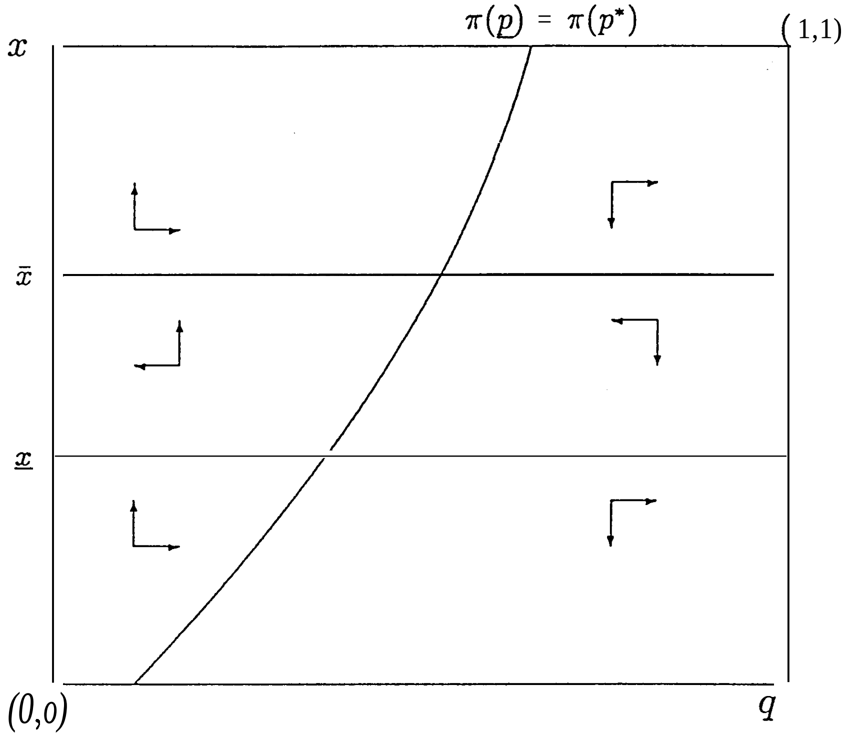

Figure 1: A simple example

If search costs are not too high, then we have the structure of equilibrium illustrated in Figure 1. There are two values of x, $\{\underline{x}, \bar{x}\}$ for which consumers are indifferent between searching once and searching twice. The variance of prices, and hence the expected return to search, is at a maximum at $x = 0.5$. At $x = 1$ or 0, searching once dominates searching twice. The curve maps the equal profit line between the sellers' two possible strategies. It is upward sloping because as q rises the profits of firms charging p^* rises. In other words,

$$d\pi(\underline{p})/dx - d\pi(p^*)/dx < 0. \tag{4}$$

Profits can only be kept equal if the number of low-priced firms x increases, which reduces the number of customers at the high-priced firms.

There are two interior equilibria, and one at the bottom right hand corner (the no-search outcome). The arrows are generated by the simple assumption that the proportion of the population playing the strategy which is earning a higher payoff will grow at the expense of the other. For example, near the no-search equilibrium, searching once is more profitable than searching twice, and p^* more profitable than $\underline{p}$. Hence the arrows point right and down. Both the interior equilibria will be unstable under any such simple dynamic. This is simply because the equal profit curve is upward-sloping. Looking at (3), it is clear that the profits of the firms charging p^* are increasing in their population share 1 — x. There is a positive externality in between these sellers in that the more of them that there are, the less the probability of consumers finding a better price. Hence, a deviation from equilibrium which, for example, increases the market share of the high-priced firms, (a shift downwards in the Figure), increases their profits. Furthermore, because of the condition (4) their profits increase more than the profits of the low-priced firms, leading to a deviation of increased size.

How would this model change if some consumers took more than two price quotations, which might happen in the noisy sequential search model? Certainly, we can no longer represent the situation graphically as in Figure 1, however, the externality would still be there. A decrease in x would still raise profits of both low-priced and high-priced sellers. Nonetheless, it is possible that $d\pi(\underline{p})/dx - d\pi(p^*)/dx < 0$. That is, the biggest rise in profits would go to the low-priced sellers. Hence, in contrast to the previous case, stability is possible.

This is of course a very simple analysis and hence possibly misleading. It might be argued, for example, that the instability in the case of a fixed sample size rule arises solely from the inadequate nature of the discrete approximation of a continuous strategy space. The positive externality shared between the highest-priced firms depends on the existence of positive mass of firms charging that price, something that may not occur when the strategy space is a continuum. For that reason we will examine the dynamics when sellers can choose from a continuum of prices. We find that it is still the case that the dispersed price equilibrium is unstable. However, the first step is to discuss the exact structure of dispersed price equilibrium in such circumstances.

3 Evolutionary Market Dynamics

Having described some possible equilibria, we now deal with disequilibrium. We imagine the above one-shot game is repeated many times. That is, at each point in time firms must choose prices and consumers a search strategy, for example, how many price quotations to buy or a reservation price to search for. As is common in the literature on learning and evolution, agents do not play some complex intertemporal equilibrium. Instead they adjust their play of the stage game. In this context, firms change prices in the direction of increasing profits. This is not unreasonable in the context of the model. Firms are assumed "small" relative to the size of the market and have little strategic power. Consumers participate in this market only infrequently. This also means that firms have little incentive to build a reputation.

The parameter μ now represents the volume of the flow of consumers on to the market. This rate is fixed and exogenous.[4] We do not assume that consumers know

[4] Fershtman and Fishman (1992) consider the case where rational forward-looking consumers may decide to defer consumption if they expect prices to fall.

the distribution of prices. Instead we assume a type of social learning of the type for example set out in Young (1993). That is, consumers have some access to information on how consumers have behaved in the past and how successful were the different strategies pursued. They obtain this information either from their own past experience of the market or from advice from more recent participants. Thus consumers act adaptively using past observations to form an estimate of the current distribution of prices. But there also needs to be some rule by which they decide on what response they make. One alternative would be for consumers to choose a search strategy which was a best response to the estimated distribution. This would be consistent with the idea of fictitious play perhaps the most popular learning model in the recent literature, (see for example, Milgrom and Roberts, 1991; Young, 1993; Fudenberg and Kreps, 1993).

However, there are other models of learning. It is possible to use the evolutionary replicator dynamics or their generalisations as a way of modelling human learning behaviour, for example, Friedman (1991), while another alternative is the learning model considered by Roth and Erev (1995), or the learning by imitation model of Schlag (1994). In an earlier paper (Hopkins, 1995), one of us demonstrated that the aggregation across a population of players of all these learning rules were qualitatively similar. For example, evolutionarily stable strategies (a definition follows) are asymptotically stable for all dynamics of this class. It is possible therefore to work with the class of positive *definite* dynamics which include all these models as special cases. Within this framework, firms and consumers could have different learning rules, or behaviour could vary within the two populations; for example, some agents could play mixed strategies, some could play pure.

Much of the work on learning and evolution has been in the context of a large population of agents who are randomly matched to play a normal form game. Unfortunately this does not match a description of most markets, at least as traditionally modelled by economists. First, agents interact not by random matching but through the price mechanism. For example, in Cournot type competition, rather than being matched one-to-one, firms interact through the effect their decisions on output have on aggregate output and hence on price. Second, profits are non-linear in the firm's decision variable. Lastly, the strategy space is a continuum.

The first difficulty is easily overcome. Even in a random matching environment the aggregate is important because it determines each agent's expected payoffs. In the case of a symmetric normal form game with n strategies, let A be the $n \times n$ payoff matrix, and let each agent play a strategy (possibly mixed) $y \in S_n = \{y = (y_1,\ldots,y_n) \in R^n : \Sigma y_i = 1, y_i \geq 0$ for $i = 1,\ldots, n\}$. If mixed strategies in the population are described by a distribution function F on S_n, then let $x \in S_n = \int y \, dF$ be the vector of the average propensity in the population to play each strategy (If all agents play a pure strategy, then x is simply the vector of proportions of the population following each strategy). Then, an individual playing the strategy y against a population with current state x, has expected payoff $y \cdot Ax$.

A positive definite dynamic is a dynamic which has the form,

$$\dot{x} = Q(x)\ Ax. \tag{5}$$

where Q is a symmetric semi-positive definite matrix. There are certain other conditions which are set out in Hopkins (1995). Similar conditions with modifications to allow for an infinite number of strategies appear in Section *4* of this paper. However, the most significant condition is simply that of positive definiteness. This ensures that the vector of changes in strategy frequencies $\dot{x}$ is at less than a 90° angle to the vector of payoffs *Ax*. This is thus a very weak formulation of the assumption that strategies with a high payoff grow at the expense of those with a lower return.

We can extend these dynamics to the case where profits are given by a nonlinear function. However, we again assume that agents choose between *n* strategies. Th return to each strategy given the state of the population x is $\pi(x) = (\pi_1, ..., \pi_n)$. Then an agent playing any strategy $y \in S_n$ would receive a payoff $y \,.\, \pi(x)$. We assume the dynamics to be given simply by

$$\dot{x} = Q(x)\pi(x). \tag{6}$$

The motivation for this approach is evolutionary. In evolutionary game theory an *evolutionarily stable strategy* (ESS) is defined as a strategy which is "uninvadable". Agents playing some alternative strategy would not be able to supplant agents who stick to the original strategy. In the market games considered in this paper, in a similar way, we want to know whether a given equilibrium distribution of prices can resist any deviation by any firm or group of firms from their equilibrium strategy. The conditions for a state ϕ to be an ESS are firstly, that ϕ should be a best reply to itself, that is, a Nash equilibrium, or formally,

$$\phi \cdot \pi(\phi) \geq x \cdot \pi(\phi),\ \forall x \in S_n. \tag{7}$$

Second, if there are any alternative best replies, any strategy x for which x . ~(#) = $\phi \,.\, \pi(\phi)$, then ϕ is better against them than they are against themselves.

$$\phi \,.\, \pi(x) > x \,.\, \pi(x). \tag{8}$$

What this last condition implies is a kind of concavity of the payoff function. If for example π were linear in x, and ϕ was a fully- mixed equilibrium $(\phi_i > 0\ \forall\ i)$, *(8)* would become $(x - \phi) \cdot \pi(x - \phi) < 0$, (as $(x - \phi) \cdot \pi(\phi) = 0$). Now as both x and ϕ are vectors summing to one, $(x - \phi)$ is an element of R_0^n, that is the space $\{x \in R^n : \sum x_i = 0\}$. Thus, evolutionary stability implies (in most cases) that a linear profit function, such as for a normal form game, must be negative definite on R_0^n. Here, with nonlinear profit functions, what we require is that the linear approximation at the equilibrium point be negative definite. That is, if $\Pi_x = d\pi/dx$, then x . $\Pi_x x < 0$, $\forall$ x $\in R_0^n$. In fact, such negative definiteness is a sufficient condition for dynamic stability under positive definite dynamics.

Proposition 1 *An equilibrium point+ is asymptotically stable under positive definite dynamics if $\Pi_x(\phi)$ is negative definite on R_0^n and asymptotically unstable if $\Pi_x(\phi)$ is positive definite.*

This proposition is a special case of Proposition 3 which, together with its proof, can be found in the next section, in which evolutionary market dynamics are examined when sellers can choose from a continuum of prices.

4 Dynamics on an Infinite Space

As Burdett and Judd themselves suggested

> "Examples of further possible work include stability analysis which may give further information concerning the durability of equilibrium price dispersion and reduce the multiplicity of equilibria in the nonsequential model." (1983, p967)

In this section, we do indeed carry out a stability analysis, both for fixed sample size model of Burdett and Judd and their model of noisy sequential search. In the former case, we will go on to analyse the dynamics the case where the two types of agents, firms and consumers, change their behaviour simultaneously. For the moment, however, we consider only the behaviour of sellers, treating consumer behaviour as fixed. We give this priority at it is likely that consumers adjust their behaviour much more slowly than do firms. Second, the problem for consumers is largely unstrategic, in that a consumer's payoffs are not affected by the actions of the other consumers. As we have seen in the simple example of Section 2, the stability of equilibrium is largely determined by the adjustment process of the sellers.

While before we used a vector x to describe the state of the population of firms, now its role is taken by a density function f. To simplify things somewhat in the non-sequential case, we normalise p^* to 1 and in the noisy sequential case, we similarly normalise $\bar{p}$. Thus we consider distributions of prices on the interval [0,1]. As for a dynamic on a finite dimensional space, we take a linear approximation to the nonlinear dynamics at the equilibrium distribution and we find that, in the first case, this equilibrium is positive definite and hence unstable. In the second, that is when consumers search sequentially, the results are less clear cut. However, we are able to show that there is at least one dispersed price equilibrium which is stable.

This is not the first attempt to examine evolutionary dynamics with a continuous strategy space. Hofbauer and Sigmund (1990), for example, note that, unlike in finite dimensions, it is possible to have evolutionary stability without dynamic stability and vice versa. We do not attempt here to obtain any general results about the links between the two concepts. However, we have already seen that when the payoff function is non-linear, the condition (S) for evolutionary stability is not identical to the condition for dynamic stability, that is, that the linear approximation Π_x is negative definite.

Let E be a complex Hilbert space, that is a Banach space with the addition of a scalar product $\langle f, g\rangle : E \times E \longrightarrow C$. An example is C^n, with the vector inner product $f \cdot \bar{f}$, where the bar indicates the complex conjugate. However, we will in particular be interested in the space $\mathcal{L}_2[0,1]$, that is, the space of (Lebesgue) measurable functions on the unit interval bounded in the norm $\|f\| = \left(\int f\ \bar{f}\, dp\right)^{1/2}$ and with inner product $\langle f, g\rangle = \int f\bar{g}\, dp$. The advantage of working with Hilbert space is that it is possible to recreate on function space many of the results obtained in finite dimensions using matrix algebra. Functions replace vectors, continuous linear operators replace matrices, a few extra assumptions have to be made, but otherwise much is the same. We note that $\mathcal{L}_2[0,1]$ possesses the following two orthogonal subspaces. Let E_0 be the subspace of functions f such that $\langle f, 1\rangle = \int f\ dp = 0$. Let E_1 be the space of constant functions. Note that $\langle f, g\rangle = 0,\ f \in E_0,\ g \in E_1$ and, as E_0 is closed, that $E = E_0 \oplus E_1$.[5] Of course we will be particularly interested in density functions, that is, elements of $S_{\mathcal{L}} = \{f \in \mathcal{L}_2[0,1] : f \text{ is real valued}, f \geq 0, \text{and } \int f\ dp = 1\}$. We will, however, wish to consider dynamics on subspaces of E. If T is some closed subset of $[0,1]$ - and in particular we will be looking at the interval $[\underline{p}, p^*]$ - let E_T be the elements of E which are zero outside T, let E_{T1} be the elements of E constant on T, and let $E_{T0} = E_0 \cap E_T$ be the elements of E_T for which $\int_T f\, dp = 0$. Clearly, E_{T0} and E_{T1} are orthogonal, with E_{T0} closed in E, so that $E = E_{T0} \oplus E_{T1}$.

We now can examine the dynamics for a continuum of prices. At any time the distribution of prices is described by $f \in S_{\mathcal{L}}$. Let $F(p) = \int_0^p f(r)\, dr$. When not in equilibrium, firms adjust prices according to a dynamics of the form

$$\dot{f} = Q(f)\pi_f \tag{9}$$

where $Q(f)$ is a continuous linear operator on E, and π_f is the profit function given by (3). Let $T = T(f)$ be the support off, then we assume that $Q(f)$ has the following properties:

1. $Q : S_{\mathcal{L}} \longrightarrow Lin[E]$ is continuous, where $Lin[E]$ is the space of continuous linear operators on E.

2. $Q(f)$ is compact.

3. $Q(f)$ is self-adjoint, that is $\langle Q(f)g, h\rangle = \langle g, Q(f)h\rangle$.

4. $Q(f)$ maps $E \longrightarrow E_{T(f)0}$.

5. $Q(f)g = 0, \forall g \in E_{T(f)1}$.

[5] Lang(1993), Corollary V.1.8

6. $Q(f)$ is *uniformly positive definite* elsewhere, i.e. there exists a constant $m > 0$ such that $(Q(f)g, g) \geq m \operatorname{Var}_f(g) > 0$, $g \notin E_{T(f)1}$, where $\operatorname{Var}_f(g)$ is the variance of g with respect to the distribution f: $\operatorname{Var}_f(g) = \int_0^1 (g - \bar{g})^2 f(p)dp$, where $\bar{g} = \operatorname{mean}_f(g) = \int_0^1 g(p)f(p)dp$.

7. If $\{f_n\} \subset S_{\mathcal{L}}$ is a sequence of continuous density functions, and p is a point for which $\lim_{n\to\infty} f_n(p) = 0$, then $\lim_{n\to\infty} \dot{f}_n(p) = 0$.

The long list of conditions should not hide the generality of the dynamics specified. The substantive conditions are numbers 3 and 6. As noted in the previous section, positive definiteness ensures that population shares of the different strategies grow more or less in line with current payoffs. Property 4 ensures that $\int \dot{f}\, dp = 0$ and hence $\int f\, dp$ continues to be equal to one. Property 5 means that a mixed Nash equilibrium, that is when all strategies have the same return, is an equilibrium for the dynamic. Properties 4 and 5, together with the decomposition $E = E_{T(f)0} \oplus E_{T(f)1}$, imply that

$$Q(f): E_{T(f)0} \to E_{T(f)0}$$

is an isomorphism. Property 7 implies that the dynamic is invariant on $S_{\mathcal{L}}$. More specifically, no strategy present in the initial distribution will disappear in finite time, nor will any new strategy be created. Thus we will want to look at cases where all prices are present in the initial distribution. This may seem somewhat restrictive, but it should be remembered that any distribution, including those where all firms charge the same price, can be approximated arbitrarily closely by a distribution with full support. Second, this formulation does not prevent the limit of the dynamic process being a state like the no-search outcome, where all sellers charge the same price.

Property 2 has two important consequences. First, unlike in finite dimensions, the spectrum of an operator on a space such as $\mathcal{L}_2$ may include elements which are not eigenvalues. However, the spectrum of a compact operator consists of its eigenvalues alone (together with zero if the space is infinite dimensional). Second, the Hilbert-Schmidt theorem (see for example, Hutson and Pyre, 1980), states that the eigenfunctions of a compact, self-adjoint operator form an orthogonal basis for E. Since one eigenspace of $Q(f)$ is $E_{T(f)1}$, the others span $E_{T(f)0}$.

A concrete example of such an operator is given by the replicator dynamics, which have the form (9) with Q given by

$$Q(f)g = f\,[g - \langle f, g\rangle]. \tag{10}$$

It is easily verified that the operator $Q(f)$ satisfies the above conditions.

We begin with two preliminary general results:

Lemma 1 *If Q is a compact linear operator and A is a continuous linear operator then QA is compact. (Lang, 1993; Theorem XVII. 1.2).*

If A is a continuous operator, then it is positive definite when constrained to E_{T0} if $\langle Af, f\rangle > 0 \ \forall \ f \in E_{T0}$ and negative definite if $\langle Af, f\rangle < 0$.

Proposition 2 *If A is positive (negative) definite on E_{T0} then QA has only positive (negative) eigenvalues when constrained to E_{T0}.*

Proof: We have the eigenvalue equation $QAf = \lambda f$ where $f \in E_{T0}$. It follows from Properties 1, 2 and 3 above, that Q : $E_{T0} \rightarrow E_{T0}$ *is* an isomorphism. Thus, we can find a (unique) g $\in E_{T0}$ such that $f = Qg$.

$$\begin{aligned} \lambda \langle f, g\rangle &= \langle QAf, g\rangle \\ \lambda \langle Qg, g\rangle &= \langle Af, Qg\rangle \\ &= \langle Af, f\rangle \end{aligned} \tag{11}$$

As Q is positive self-adjoint on E_{T0}, $\langle Qg, g\rangle$ is a positive real number. By hypothesis, the real part of $\langle Af, f\rangle$ is positive (negative), hence all eigenvalues λ for eigenfunctions of QA in E_{T0} have real part positive (negative). Furthermore, from Lemma 1 , all its spectrum is positive (negative). • 1

We can write the firms' profits as $\pi_f = \Pi(f)$. That is, $\Pi : S_{\mathcal{L}} \rightarrow E$ is a (non-linear) operator mapping a distribution of prices f into a distribution of profits π_f given by (3). It is possible to perform calculus in function spaces such as Hilbert spaces, (see Lang, 1993; Hutson and Pyre, 1980). In particular, the operator Π is differentiable at $\phi \in S_{\mathcal{L}}$ if there exists a (necessarily unique) linear operator $\Pi'_\phi : E \rightarrow E$ such that

$$\lim_{\|f\| \rightarrow 0} \left\| \Pi(\phi + f) - \Pi(\phi) - \Pi'_\phi f \right\| / \|f\| = 0.$$

In this case, we have

$$\Pi'_\phi z = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\Pi(\phi + \varepsilon z) - \Pi(\phi)\right] \quad (a.e.)$$

Looking at the profit function (l), we can see that it is continuously differentiable in F. If we calculate the linearisation of the profit function around the equilibrium distribution ϕ, we have

$$\Pi'_\phi f(p) = -p\mu F(p) \sum_{k=2}^{\infty} q_k k(k-1)[1 - \Phi(p)]^{k-2} < 0 \quad . \tag{12}$$

Profit is decreasing in the number of competitors charging a lower price. In the fixed sample size model of Burdett and Judd, firms' profits are an affine function of the distribution of prices (q_k = O, for $k > 2$). Differentiation simply removes the '(intercept" term leaving,

$$\Pi'_\phi f = -2p\mu(1 - q_1)F. \tag{13}$$

More generally, a continuous (non-linear) operator, $Q : S_{\mathcal{L}} :\rightarrow Lin\,[E]$, is differentiable at ϕ if there is a continuous *bilinear* operator, $Q'_\phi : E \times E \rightarrow E$, satisfying

$$\lim_{||f||\rightarrow 0} ||Q(\phi+f)g - Q(\phi)g - Q'_\phi(f,g)||/||f|| = 0, \quad \forall g \in E$$

and in this case, we have

$$Q'_\phi(z,g) = \lim_{\varepsilon\rightarrow 0}\frac{1}{\varepsilon}\left[Q(\phi+\varepsilon z)g - Q(\phi)g\right] \quad (a.e.)$$

We shall assume that the operator Q defining the positive definite dynamics (9) is differentiable (of course, this is stronger than the continuity assumption in Property 1 above). In particular, the replicator dynamic operator (10) is differentiable, with

$$Q'_\phi(g,z) = zg - \langle\phi,g\rangle z - \langle z,g\rangle\phi$$

We now consider the linearisation of the dynamics (9) in a neighbourhood of an equilibrium $\phi \in S_{\mathcal{L}}$. The theory of Hartman and Grobman that a non-linear differential equation is locally equivalent to its linear part at a hyperbolic fixed point is valid on Banach (and hence Hilbert) spaces (Palis and de Melo, 1982). Hence, if we write $f = \phi + \varepsilon z$, with $z \in E_0$, in a neighbourhood of ϕ, we can approximate the non-linear equation (9) by

$$\dot{z} = Q'_\phi(\pi_\phi, z) + Q(\phi)\Pi'_\phi z, \quad z \in E_0. \tag{14}$$

Let $T = [\underline{p}, 1]$ be the support of ϕ, and consider the orthogonal decomposition, $E = E_{T0} \oplus E_{T1}$. Thus, $f \in E$ may be written uniquely in the form $f = f_0 + f_1$, where $\langle f_0, f_1\rangle = 0$. In fact,

$$\begin{aligned} f_0(p) &= \begin{cases} 0 & \text{for } p \in [0,\underline{p}) \\ f(p) - \frac{F(1)-F(\underline{p})}{1-\underline{p}} & \text{for } p \in [\underline{p},1] \end{cases} \\ f_1(p) &= \begin{cases} f(p) & \text{for } p \in [0,\underline{p}) \\ \frac{F(1)-F(\underline{p})}{1-\underline{p}} & \text{for } p \in [\underline{p},1] \end{cases} \end{aligned} \tag{15}$$

In particular, for $z \in E_0$, we can write $z = z_0 + z_1$, with $z_0 \in E_{T0}$ and $z_1 \in E_{T01} = E_0 \cap E_{T1}$, and the local dynamics (14) decomposes into two corresponding parts.

We can now state our main results.

Proposition 3 *If Π'_ϕ, the linearisation of the profit function taken at an equilibrium distribution ϕ with support T, is a negative definite linear operator on E_{T0} then ϕ is locally asymptotically stable under the positive definite dynamics (9) on E_T. If it is positive definite, ϕ is asymptotically unstable on E_T.*

Proof: For the dynamics on E_T we are concerned with perturbations from equilibrium of the form $f = \phi + \varepsilon z$, with $z \in E_{T0}$ (i.e. $z_1 = 0$). Now, since ϕ is an equilibrium distribution, $Q(\phi)\pi_\phi = 0$, so that $\pi_\phi \in \mathrm{Ker} Q(\phi) = E_{T1}$; i.e. π_ϕ is constant on T. Thus, $Q(f)\pi_\phi = 0$ for any $f \in E_T$ by Property 5, from which it follows that $Q'_\phi(\pi_\phi, z) = 0$, and (14) simplifies to

$$\dot{z} = Q(\phi)\Pi'_\phi z, \quad z \in E_{T0} \tag{16}$$

By Proposition 2, this linearisation has only negative (positive) eigenvalues if Π'_ϕ is negative (positive) definite. In either case the linearisation is hyperbolic (there are no eigenvalues with real part zero). The solution to the linear equation (16) is given by $\exp(tQ\Pi')z^0$, where z^0 is an arbitrary initial distribution of prices. (The exponent of a linear operator tL is given by the polynomial $\sum_{k=0}^{\infty}(tL)^k/k!$. Since the normed vector space of bounded linear operators on $\mathcal{L}_2$ is a Banach space, it is complete and hence the limit of this convergent series is itself an operator on $\mathcal{L}_2$.) Furthermore, $\exp(tQ(\phi)\Pi')$ has the same eigenfunctions as $tQ(\phi)\Pi'$ and eigenvalues $\exp(\lambda t)$. $Q\Pi'$ is compact because Q is compact and Π' is linear (Lemma 1). By the spectral theorem for compact operators (Lang, 1993; Theorems XVII 3.4, 3.5), we can make a direct sum decomposition of E_T into the (generalised) eigenspaces of $Q\Pi'$. Thus, just as in the finite dimensional case, all negative eigenvalues implies asymptotic stability, all positive asymptotic instability on ET. 1

Proposition 4 *Under -the positive definite dynamics (9), a dispersed price equilibrium is unstable when consumers use a fixed sample size rule.*

Proof: Given an equilibrium density of prices ϕ with support T, we consider a perturbed distribution of prices,

$$f = \phi + \varepsilon z, \qquad z \in E_{T0}, \ ||z|| > 0.$$

Let $Z(p) = \int_0^p z\, dr$. Of course, $Z(0) = Z(1) = 0$. As f is arbitrary, if the quadratic form $\langle \Pi'_\phi z, z\rangle$ is positive then Π'_ϕ is positive definite on E_{T0}.

$$\begin{aligned} \langle \Pi' z, z\rangle &= \textstyle\int_T z\,.\Pi' z\ dp \\ &= \textstyle\int_T \Pi' z\ dZ \end{aligned}$$

Using (13), we have

$$\int_T \Pi' z\ dZ = -2\mu q_2 \int_T pZ\, dZ,$$

which, on integration by parts gives

$$\mu q_2 \int_T Z^2\, dp > 0.$$

Hence the linearisation (13) is positive definite and by Proposition 3 the equilibrium is unstable on ET. *1*

Of course, even if an equilibrium is unstable only on a subset of the total state space, $E_T \subset E$, it is still described as unstable. However, if we allow $q_k > 0$ for $k > 2$, then the above reasoning only shows that the sign of the expression $\langle \Pi'_\phi z, z \rangle$ is ambiguous. Nevertheless, we can derive conditions for the linearisation to be negative definite on *ET, as* follows.

From (3), the profit to firms at equilibrium is $\pi^* = \pi_\phi(1) = \mu q_1$, and this is positive provided $q_1 > 0$, which we will assume. This profit must be the same for all prices charged at equilibrium, in particular, $\pi_\phi(\underline{p}) = \pi^*$, which gives

$$\underline{p} = \frac{\pi^*}{\mu \bar{k}} = \frac{q_1}{\bar{k}}$$

where $\bar{k}$ = ~& $k q_k$ *is* the mean sample size. Clearly, $0 < \underline{p} \leq 1$, with $\underline{p} < 1$ if $q_1 < 1$, which we will also assume. More generally, since ϕ has support $[\underline{p}, 1]$, we have $\pi_\phi(p) = \pi^*$ for all $p \in [\underline{p}, 1]$, and (3) gives

$$p \sum_{k=1}^{\infty} q_k k \left[1 - \Phi(p)\right]^{k-1} = \frac{\pi^*}{\mu} \tag{17}$$

for $p \in [\underline{p}, 1]$. Now $\Phi(p)$ is strictly increasing from 0 to 1 as p increases from $\underline{p}$ to 1. Hence, $\Phi : [\underline{p}, 1] \to [0, 1]$ is an isomorphism, and we can write $p = \Phi^{-1}(x)$ for a unique $x \in [0, 1]$.

Write g(x) = ~& $q_k k(1 - x)^{k-1}$, for $x \in [0, 1]$. Then, from (17), P = ‘$^{-1}$(z) = $\pi^*/\mu g(x)$. Let $T = [\underline{p}, 1]$ and $z \in E_{T0}$. Then, from (12)

$$\Pi'_\phi z(p) = \begin{cases} 0 & \text{for } p \in [0, \underline{p}] \\ \pi^* \frac{g'(x)}{g(x)} Z(p) & \text{for } p \in [\underline{p}, 1] \end{cases} \tag{18}$$

[Recall that $z \in E_{T0}$ implies that $z(p) = 0$ for $p \in [0, \underline{p}]$.] Thus

$$\langle \Pi' z, z \rangle = \int_{\underline{p}}^{1} \Pi' z dZ = \frac{\pi^*}{2} \int_{\underline{p}}^{1} \frac{g'(x)}{g(x)} dZ^2$$

Integrating by parts and using $Z(\underline{p}) = Z(1) = 0$ gives

$$\langle \Pi' z, z \rangle = -\frac{\pi^*}{2} \int_{\underline{p}}^{1} \frac{d}{dp} \left[\frac{g'(x)}{g(x)}\right] Z^2 dp = -\frac{\pi^*}{2} \int_{0}^{1} \frac{d}{dx} \left[\frac{g'(x)}{g(x)}\right] Z^2 dx \tag{19}$$

It now follows immediately that Π' is negative definite on E_{T0} if and only if

$$g(x) g''(x) - g'(x)^2 > 0 \quad \text{for all } x \in [0, 1] \tag{20}$$

We can show that there exist equilibria which satisfy (20). By Proposition 3, such equilibria are stable on E_T. However, more than this, we can also show that they are locally asymptotically stable on the whole of $S_{\mathcal{L}}$. This involves consideration of the local dynamics (14) for perturbations $z \in E_{T1}$. Such perturbations allow for the possibility that there is a non-zero density of firms which undercut the lowest equilibrium price $\underline{p}$. The details will be given in Appendix 1. Thus, we can prove:

Proposition 5 *There exist dispersed price equilibria which are locally asymptotically stable under the positive definite dynamics (9).*

We can place an interpretation on these two results. Firstly, it is important to realise that one consequence of (12) being negative is that it is possible to construct deviations from equilibrium which raise the profits of all sellers (less a set of measure zero). Imagine an alternative distribution f which places greater weight on high prices: $z = f - \phi$ would be negative for low prices and positive nearer 1. Assume $f < \phi$ on the interval $(\underline{p}, a)$, $f(a) = \phi(a)$, and $f > \phi$ on the interval (a, 1). Because, $F < \Phi$ except at the two end points of the distribution and because profits are decreasing in F, profits are higher everywhere. For example, in the fixed sample size model, profits are

$$p[q_1 + 2q_2(1 - F)] \geq p[q_1 + 2q_2(1 - \Phi)],$$

as $F \leq \Phi$ on $(\underline{p}, 1)$. The change in profits is obviously equal to $p\,2q_2(\Phi - F)$, and thus is increasing in price and the difference between the two cumulative distributions. Profits are unchanged at $p = 1$ as both F and Φ must be equal to one at this point. The greatest increase in profits will occur at $p = (\Phi - F)/(2q_2(f - \phi)) > a$, that is, at a point where the density of firms has increased. Just as for the simple example of Section 2, an increase in density of firms at certain prices leads to an increase in profits for all firms charging that price. Under positive definite dynamics, this will result in the deviation from equilibrium increasing in magnitude, and is obviously destabilizing. There is nothing special about this example. Since all eigenvalues of $Q\Pi'$ (when constrained to E_{T0}) are positive, all deviations have a similarly destructive effect.

Another way of writing (19) is as $-\int g'(1+\eta)Z^2\,dp$, where $\eta = (p/g')\,(dg'/dp)$. If we consider g' as the marginal demand faced by the sellers, then η is the price elasticity of that demand. In the model considered above, $\eta = 0$. However, in the case where $q_k > 0$ for $k > 2$, which may hold for equilibria under noisy sequential search, η is negative. Because g' remains negative, a deviation similar to the one considered above will still raise profits for all sellers where $F < \Phi$. The important difference is that the maximum change in profit may occur on the interval $(\underline{p}, a)$. If indeed $dg'/dp > 0$, the biggest increase in profits from the deviation from equilibrium falls to the firms that have kept their prices low. Clearly, the more price information that consumers have, the greater is the price elasticity that sellers face. If the marginal demand is sufficiently elastic (a sufficient though not necessary condition would be for $\eta < -1$, *a.e.*), the number of firms charging low prices will grow and the distribution of prices will return to equilibrium.

5 Consumer Dynamics

Having considered dynamics for sellers, we now examine what happens when buyers and sellers change behaviour simultaneously in the fixed sample size case (in the model of noisy sequential search, the distribution $\{q_k\}$ is a parameter, and is not decided by consumer choice, and so cannot be the subject of dynamic analysis). Unfortunately, we have been unable to derive results in infinite dimensions for the dynamical system representing simultaneous learning by buyers and sellers. Hence we use a discrete approximation to the continuous distribution. Obviously though, this approximation can be arbitrarily close to the original.

It is important to remember that the behaviour of other buyers does not enter directly into the decision of any individual consumer. The payoff to each consumer is determined by his decision on how much and how to search and the current distribution of prices, not by the search behaviour of other consumers. Of course, there may be an indirect effect. For example, if average consumer search is very intense, then there will be a downward pressure on prices, which will in turn change consumers' expected payoffs. But it remains the case that the dynamic stability of a dispersed equilibrium is largely determined by whether there is stability in the adjustment process for sellers. We use this fact to prove the following proposition.

Proposition 6 When *the changes of both buyer and seller behaviour are described by positive definite dynamics, the discrete approximation to the mixed equilibrium of the model of Burdett and Judd (1983) is unstable.*

Proof: In Appendix 2.

6 Discussion

We have shown that, in the model of Burdett and Judd (1983), dispersed price equilibria are unstable when consumers use a fixed sample size rule. In contrast, when a sequential search rule is used, the adjustment process for sellers may converge to a dispersed price equilibrium. In other words, taking an evolutionary approach has enabled us, first of all, to discriminate between different equilibria on the basis of stability, and secondly, to discriminate between models. That is, the model where consumers adopt a fixed sample size search rule does not possess a stable dispersed price equilibrium. Thus, we would argue, it does not provide a good theoretical basis for explaining the dispersion of prices, when compared with the model with noisy sequential search.

What this paper has omitted is an investigation of how in the sequential search case, consumers choose their reservation price. This is something for further research, but it is worth noting the following. It is not known the way that real consumers search, but there are reasons to believe that this search does not take the form of a fixed reservation price rule. This is despite the fact that this has become the dominant paradigm in economic theory. Firstly, as Morgan and Manning (1985) show, the optimal search rule in many cases will take the form of a mixture between a fixed sample size and a sequential rule. That is, the searcher immediately obtains several quotations but then may take more if the offers received are unsatisfactory. Harrison and Morgan (1990) find that behaviour under experimental conditions fits this pattern. Second, as Telser (1973), Rothschild (1974) and Gastwirth (1976) all find, the optimality of a reservation price rule does not withstand the introduction of imperfect information. If one fails to calculate the reservation price correctly, it is possible to search for an arbitrarily long time without success. Or as Telser puts it "if the searcher is ignorant of the distribution, then acceptance of the first choice drawn at random from the distribution confers a lower average cost than more sophisticated procedures for a wide range of distributions" (1973, p45). Thus in various senses a fixed reservation price rule does not seem robust. It is therefore unlikely that it would form the endpoint of an adaptive process. A direction for further research would be to see whether the evolutionary approach can pick out simple and robust search rules.

Appendix 1

Proof of Proposition 5. The proof has two stages. First, it is shown that there exists a dispersed price equilibrium which is negative definite on its support T, and, second, that if the system is close to this equilibrium but with full support, it approaches the equilibrium.

For the first part, we must find a distribution, $q = \{q_k\}$, for which (20) holds. To keep things as simple as possible, take $q_k = 0$ for $k > 3$. Then

$$\begin{aligned} g(x) &= q_1 + 2q_2(1-x) + 3q_3(1-x)^2 \\ g'(x) &= -2q_2 - 6q_3(1-x) \\ g''(x) &= 6q_3 \end{aligned}$$

Thus, condition (20) reduces to

$$(3q_1q_3 - 2q_2^2) - 6q_2q_3(1-x) - 9q_3^2(1-x)^2 > 0 \quad \text{for all } x \in [0,1].$$

This holds if and only if it holds for $x = 0$; i.e. if and only if

$$(3q_1q_3 - 2q_2^2) - 6q_2q_3 - 9q_3^2 > 0.$$

Substituting for $q_1 = 1 - q_2 - q_3$ gives

$$2q_2^2 + 9q_2q_3 - 9q_3(\frac{3}{4} - q_3) < 0. \tag{21}$$

Fix q_3 in the range $0 < q_3 < \frac{3}{4}$. Then (21) holds for all $0 < q_2 < \bar{q}_2$, where

$$\bar{q}_2 = \frac{1}{4}\{-9q_3 + [9q_3^2 + 54q_3]^{\frac{1}{2}}\} > 0.$$

We also require $q_2 + q_3 < 1$ (which implies that $0 < q_1 < 1$). This holds if $\bar{q}_2 + q_3 < 1$; i.e. if

$$[9q_3^2 + 54q_3]^{\frac{1}{2}} < 5q_3 + 4$$

But this reduces to the condition, $8q_3^2 - 7q_3 + 8 > 0$, which holds for all real q_3.

This completes the construction of a distribution (in fact, infinitely many such), $q = \{q_1, q_2, q_3\}$, satisfying (21), and hence (20). We conclude from Proposition 3 that the corresponding equilibrium ϕ is locally asymptotically stable on E_T.

It is worth noting at this stage that, if we take $q_k = 0$ for $k > 2$, then $g''(x) = 0$, so that condition (20) cannot possibly be satisfied. This is the scenario covered by Proposition 4.

To complete the proof of Proposition 5, we must consider perturbations from equilibrium of the form $f = \phi + \varepsilon z$, with $z \in E_{T01} = E_0 \cap E_{T1}$. For such perturbations, the local dynamics (14) does not reduce to the simpler form (16). Neither is it true that E_{T01} is invariant under the dynamics. However, we have the following simple criterion for such a perturbation to decay.

Lemma 2 *Let $\alpha = \pi^* - \pi_\phi \in E_{T1}$. Then, for $f \in E$ with $f \geq 0$ on $[0, \underline{p})$, $\langle f, \alpha \rangle \geq 0$ with equality if and only if $f \in E_T$. Hence, the component $z_1(t) \in E_{T01} \to 0$ as $t \to \infty$ if and only if $\langle z(t), \alpha \rangle \to 0$ as $t \to \infty$.*

Proof: By definition, $\pi_\phi = \pi^*$ on $T = [\underline{p}, 1]$, and $\pi_\phi(p) = \pi^* \cdot p/\underline{p}$ for $p \in [0, \underline{p})$. Thus, $\alpha = 0$ on T, and $\alpha(p) = \pi^*[1 - p/\underline{p}] > 0$ for $p \in [0, \underline{p})$. That $\langle f, \alpha \rangle \geq 0$, with equality if and only if $f \in E_T$ now follows easily from the decomposition (15) and the assumption that $f \geq 0$ on $[0, \underline{p})$. Taking $f(t) = \phi + \varepsilon z(t) \in S_{\mathcal{L}}$, it follows from (15) that $f(t) = f_1(t) = \varepsilon z_1(t) \geq 0$ on $[0, \underline{p})$. The second assertion then follows from the fact that $E_T \cap E_{T01} = \{0\}$. □

Next, note that $\langle Q(\phi)\Pi'_\phi z, \alpha \rangle = 0$, by Property 4 and Lemma 2. Thus, the dynamics (14) give

$$\frac{d}{dt}\langle z, \alpha \rangle = \langle \dot{z}, \alpha \rangle = \langle Q'_\phi(\pi_\phi, z), \alpha \rangle$$

Again, since $Q(f)\pi = 0$ for any f, and $Q(\phi)\pi_\phi = 0$, we have

$$Q'_\phi(\pi_\phi, z) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} Q(\phi + \varepsilon z)\pi_\phi = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} Q(\phi + \varepsilon z)\alpha = -Q'_\phi(\alpha, z).$$

Thus,

$$\frac{d}{dt}\langle z, \alpha \rangle = -\langle Q'_\phi(\alpha, z), \alpha \rangle \tag{22}$$

It remains to compute the angle-bracket term in (22). To do this, we exploit the uniform positive definiteness of $Q(f)$ (Property 6). Thus, an easy computation gives

$$\langle Q(\phi + \varepsilon z)\alpha, \alpha \rangle \geq m \ \mathrm{Var}_{\phi + \varepsilon z}(\alpha) = \varepsilon m \langle z\alpha, \alpha \rangle + \mathcal{O}[\varepsilon^2]$$

Hence,

$$\langle Q'_\phi(\alpha, z), \alpha \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle Q(\phi + \varepsilon z)\alpha, \alpha \rangle \geq m \langle z\alpha, \alpha \rangle$$

Substituting into equation (22) gives

$$\langle \dot{z}(t), \alpha \rangle \leq -m \langle z\alpha, \alpha \rangle$$

Now, from Lemma 2 and (15), we have $\langle z, \alpha \rangle = \langle z_1, \alpha \rangle \geq 0$, and similarly $\langle z\alpha, \alpha \rangle = \langle z_1\alpha, \alpha \rangle \geq 0$, with equality if and only if $z_1 = 0$. Thus, $\langle z, \alpha \rangle$ is non-negative and *strictly* decreasing as long as $z_1 > 0$ on some subset of $[0, \underline{p})$ with non-zero measure. It follows that $\langle z, \alpha \rangle \to 0$ as $t \to \infty$, and therefore from Lemma 2 that the component z_1 of z in E_{T01} decays to zero. Combining this with the construction of a distribution $q = \{q_k\}$for which ϕ is locally asymptotically stable on E_T (i.e. for which z_0 decays to zero), now completes the proof of Proposition 5. □

We remark that the second part of the above proof does not depend on any special properties of the distribution q, and therefore applies to the cases considered in either Proposition 4 or Proposition 5. This shows that the local asymptotic stability, or instability of an equilibrium ϕ is determined entirely by the dynamics in E_T.

Appendix 2

We assume that firms can choose from only a finite number n of prices. These we label $p = (p_1, p_2, \ldots.., p_n)$, where the elements are given in increasing order $p_i > p_{i-1}$. And let the proportion of the population of firms choosing each of those prices be $x = (x_1, x_2, \ldots, x_n)$. Define $F_i = \sum_{j=1}^{i} x_j$. The expected costs (that is, the expected price plus cost of searching) for the proportion q_1 of consumers who search once, and the q_2 who search twice are respectively

$$C_1 = -p \cdot x \text{ and } C_2 = -c - \sum p_i(x_i^2 + 2x_i(1 - F_i)). \tag{23}$$

The probability that a consumer who makes two searches finds p_i as the lowest price is $x_i^2 + 2x_i(1 - F_i)$. These consumers we assume to be equally divided between the x_i firms charging that price. Hence if q_2 consumers search twice and q_1 once, profits are given by

$$\pi_i = p_i \mu \left[q_1 + q_2(x_i + 2(1 - F_i)) \right].$$

Let

$$\dot{x} = Q(x)\pi(x, q) \text{ and } \dot{q} = Q(q)C(x) \tag{24}$$

where both $Q(x)$ and $Q(q)$ are continuously-differentiable symmetric, positive semi-definite matrix functions (the full requirements for a finite-dimensional positive definite dynamic are set out in Hopkins, 1995). Let s be the combined vector $(x, q) \in S = S_n \times S_2$. We have

$$\dot{s} = Q(s)\Pi(s) = \begin{pmatrix} Q(x) & 0 \\ 0 & Q(q) \end{pmatrix} \begin{pmatrix} \pi \\ C \end{pmatrix}. \tag{25}$$

Proof of Proposition 6. We assume that in equilibrium v $\leq n$ prices are charged. We then simply relabel these v prices, $(p_1, ..., p_n)$, resetting n = ν.[6] The linear approximation of the system of equations (25) at an equilibrium point s^* is given by $Q(s^*)\Pi'$. Given the consumers' payoffs (23), dC/dq is a matrix of zeros. Thus the matrix $Q(s^*)\Pi'$ has the form,

$$\begin{pmatrix} Q(x)\dfrac{d\pi}{dx} & Q(x)\dfrac{d\pi}{dq} \\ Q(q)\dfrac{dC}{dx} & 0 \end{pmatrix}$$

Stability in the single population case depended on whether the matrix II' was positive or negative definite on R_0^n. A $n \times n$ matrix A is positive definite on R_0^n iff the $(n-1)\times(n-1)$ matrix B is positive definite (van Damme, 1991), where B is defined by

$$b_{ij} = a_{ij} + a_{nn} - a_{in} - a_{nj}, \quad 1 \leq i,j \leq n-1. \tag{26}$$

In the two population case, if there are n strategies available to firms and m to consumers, then we want to know the sign of the eigenvalues of $Q(s^*)\Pi'$ when constrained to $R_{00}^{n+m} = \{x \in R^{n+m} : \sum_1^n x_i = 0, \sum_{n+1}^{n+m} x_i = 0\}$. To do this, we repeat the procedure outlined in (26) for each of the four submatrices of $Q(s^*)\Pi'$. We first look at $d\pi/dx$. In this case the matrix $(B+B^T)/2$ is

$$\mu q_2 \begin{pmatrix} p_n - p_1 & p_n - p_2 & \cdots & p_n - p_{n-1} \\ p_n - p_2 & p_n - p_2 & \cdots & p_n - p_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ p_n - p_{n-1} & \cdots & p_n - p_{n-1} & p_n - p_{n-1} \end{pmatrix} \tag{27}$$

If we subtract the nth column from the $n-1$th, and the $n-1$th from the $n-2$th and so on, a upper diagonal matrix is left, with a strictly positive diagonal. Thus this matrix and hence B are positive definite. Hence by Proposition 2, $Q(x)\, d\pi/dx$ has only positive eigenvalues, when constrained to R_0^n. So necessarily it has a positive trace. Since the bottom right submatrix of $Q(s^*)\Pi'$ is all zero, its trace is zero both before and after the process (26). Thus, the matrix as a whole when constrained to R_{00}^{n+m} has a strictly positive trace and hence at least one positive eigenvalue. □

[6] In confining our attention to the n prices charged in equilibrium, we are only going to be able to prove the equilibrium's instability on a subset of the total state space. But this is sufficient.

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